HYPERBOLIC DIMENSION AND RADIAL JULIA SETS
OF TRANSCENDENTAL FUNCTIONS

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ABSTRACT. We survey the definition of the radial Julia set $J_r(f)$ of a meromorphic function (in fact, more generally, any *Ahlfors islands map*), and give a simple proof that the Hausdorff dimension of $J_r(f)$ and the hyperbolic dimension $\dim_{\mathrm{hyp}}(f)$ always coincide.

1. INTRODUCTION

The study of the measurable dynamics of rational functions is well-developed. In particular, it is known [P, DU] that a large number of dynamically natural quantities coincide (compare also the survey article [U2]):

 (a) The Hausdorff dimension of the *radial* (also called *conical*) *Julia set* $J_r(f)$, introduced by Urbanski [U1] and McMullen [McM] as the set of points where it is possible to go from small to big scales via univalent iterates. (See Definition 2.6.)
 (b) The *hyperbolic dimension*, introduced by Shishikura [Sh] as the supremum of the Hausdorff dimensions of hyperbolic subsets of $J(f)$. (See Definition 2.8.)
 (c) The *dynamical dimension*, defined by Denker and Urbanski [DU] as the supremum over the Hausdorff dimensions of all ergodic invariant probability measures of positive entropy.
 (d) The minimal exponent for which a *conformal measure* in the sense of Sullivan exists.

In well-controlled cases, these numbers also agree with the Hausdorff dimension of $J(f)$, but it is unknown whether this holds in general (for transcendental functions, this is often false [S3, UZ], and results of Avila and Lyubich [AL] also suggest the existence of rational functions for which this equality fails).

In recent years, there have been a large number of cases where the above program was also carried through for transcendental entire and meromorphic functions. (We refer the reader to the survey [KU].) This suggests the question which parts of the general rational theory carries through for *arbitrary* transcendental meromorphic functions. The purpose of this note is to observe the following (where $\dim_H$ denotes Hausdorff dimension).

1.1. Theorem (Equality of hyperbolic and radial dimensions)**.** *Let* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *be any nonconstant and nonlinear meromorphic function. Then*

$$\dim_{\mathrm{hyp}}(f) = \dim_H(J_r(f)).$$

Remark. In fact, we prove this theorem even more generally for the class of *Ahlfors islands maps* studied by Epstein [E2] and Oudkerk [O] (see Definition 2.1 below).

Previously, this equality was usually established using conformal measures of the correct exponent for the function f. However, the construction of such measures seems more difficult in the transcendental case, and, in currently known cases, uses some additional dynamical and function-theoretic assumptions.

Our main observation is that it is possible to avoid this step, and construct a suitable hyperbolic set directly from the definition of the radial Julia set. Even for rational functions, this argument (though not the theorem) seems to be new. We believe the construction is interesting due to its simplicity and wide generality.

Estimating the hyperbolic dimension of f (and hence the dimension of $J_r(f)$) is more difficult. Stallard has shown that the Julia set of a transcendental meromorphic function can have any Hausdorff dimension in the interval $(0, 2]$, and that for a transcendental entire function this dimension can be any number in the interval $(1, 2]$. While she proved [S1] that the hyperbolic dimension is always strictly greater than 0 (the proof carries over directly to Ahlfors islands maps, and we sketch it in Corollary 2.12 below), it is not known whether the Hausdorff dimension of the Julia set of an entire transcendental function can equal the lower bound 1.

For functions with a *logarithmic singularity* over infinity, though, this bound cannot be achieved (this was proved originally in [S2] for entire functions in the *Eremenko-Lyubich class* $\mathcal{B}$, and subsequently strenghtened in [RS, BRS]). Recently, Barański, Karpińska and Zdunik [BKZ] extended this result to show also that the hyperbolic dimension of such a function is strictly larger than one. We discuss a minor generalization of this fact in Theorem A.1.

Structure of the article. In Section 2, we review a number of definitions which do not explicitly appear in the literature in this generality. The beautiful concepts of finite-type and Ahlfors islands maps, both due to Adam Epstein, have appeared only in the unpublished references [E1, E2, O], which are not widely available, so we take the opportunity to introduce them here. We also define radial Julia sets, whose well-known definition in the rational case carries over verbatim but does not seem to have been formally stated in the transcendental setting. We also review the definition of hyperbolic dimension, and for the reader's convenience include a short summary on an elementary type of conformal iterated function system which will be used to construct hyperbolic sets. Finally, we prove Theorem 1.1 in Section 3, and discuss hyperbolic dimension in logarithmic tracts in an appendix.

Notation. Throughout the rest of this article, X is a compact Riemann surface and $W \subset X$ is a nonempty open subset. A reader who is interested only in the classical cases

may wish to assume that $W = \mathbb{C}$ and $X = \hat{\mathbb{C}}$ (for transcendental meromorphic functions) or $W = X = \hat{\mathbb{C}}$ (for rational functions), and skip the subsection on Ahlfors islands maps in Section 2.

In what follows, all distances, balls, derivatives etc. are considered with respect to the natural conformal metric on the Riemann surface X (i.e., the unique metric of constant curvature -1, 0 or 1). Since X is compact, for sufficiently small $\delta > 0$, any disk

$$\mathbb{D}_\delta(z_0) = \{z \in X : \operatorname{dist}(z_0, z) < \delta\}$$

is simply connected with Jordan curve boundary. Recall that, by Radò's theorem, X is second countable, so there is a countable base for the topology of X consisting of such simply-connected balls.

If A is a topological space and $a \in A$, then we denote by $\operatorname{Comp}_a(A)$ the connected component of A containing a.

Finally, we remark that "meromorphic" for us always includes "entire" as a special case.

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2. Definitions

Ahlfors islands maps. The class of *Ahlfors islands maps* was suggested by Adam Epstein [E2] as a generalization of both general transcendental meromorphic functions and his "finite-type maps" (see below), and studied in greater detail by him and Richard Oudkerk [O]. This class provides a general setting for the study of transcendental dynamics. We remark that a similar condition for a restricted class of maps (those defined outside a small subset of the Riemann sphere) was considered by Baker, Dominguez and Herring in [BDH].

2.1. Definition (Ahlfors Islands maps).
Let $W \subset X$ be open and nonempty. We say that a nonconstant holomorphic function $f : W \to X$ is an *Ahlfors islands map* if there is a finite number k satisfying the following condition (known as the "islands property").

Let $V_1, \ldots, V_k \subset X$ be Jordan domains with pairwise disjoint closures, and let $U \subset X$ be open and connected with $U \cap \partial W \neq \emptyset$. Then for every component U_0 of $U \cap W$, there is $i \in \{1, \ldots, k\}$ such that f has an island over V_i in U_0. (That is, there is a domain $G \subset U_0$ such that $f : G \to V_i$ is a conformal isomorphism.)

An Ahlfors islands map is *elementary* if $W = X$ and $f : W \to X$ is a conformal isomorphism.

The set of all non-elementary Ahlfors islands maps from W to X is denoted $\mathcal{A}(W, X)$. We also set

$$\mathcal{A}(X) := \bigcup \{\mathcal{A}(W, X) : W \subset X \text{ open}\} \quad \text{and}$$

$$\mathcal{A} := \bigcup \{\mathcal{A}(X) : X \text{ is a compact Riemann Surface}\}.$$

An important feature of the class of Ahlfors islands maps is that it is closed under composition. We note that nonconstant rational functions are Ahlfors islands maps by definition. The name of this class is derived from the classical five-islands theorem [B]:

2.2. Theorem (Ahlfors Five Islands Theorem). *Every transcendental meromorphic function* $f : \mathbb{C} \to \hat{\mathbb{C}}$ *is an element of* $\mathcal{A}(\mathbb{C}, \hat{\mathbb{C}})$ *(with* $k = 5$*).*

There is also a *normal families version* [B, Theorem A.1] of Theorem 2.2, which we use a number of times below: if $\mathcal{F}$ is a family of analytic functions which have no islands over given domains $V_1, \ldots, V_n$ as above, then $\mathcal{F}$ is normal.

Another interesting subclass of $\mathcal{A}(W, X)$ is that of *finite-type maps*, first introduced in [E1, Chapter 3], which provides a natural generalization of the dynamics of rational functions and meromorphic functions with finitely many singular values. A nonconstant analytic function $f : W \to X$ is a *finite-type map* if it has only finitely many singular values in X, and furthermore there are no isolated removable singularites in ∂W.

2.3. Theorem (Epstein). *If* $f : W \to X$ *is a finite-type map, then* f *is an Ahlfors islands map.*

There are many examples of Ahlfors islands maps which do not fall into either of the previous two classes. For example. a "Valiron function" is an analytic map $f : \mathbb{D} \to \mathbb{C}$ which is unbounded in $\mathbb{D}$, but bounded on a "spiral", i.e., on a curve $\gamma(t) = r(t) \cdot e^{2\pi i \vartheta(t)}$ with $r(t) \nearrow 1$ and $|\vartheta(t)| \to \infty$. Every such function is an Ahlfors map (see [BR2, Remark in Section 4]) — indeed, these functions are not normal near any point of $\partial \mathbb{D}$, and the result follows by the normal families version of the Ahlfors five islands theorem. For the same reason, other functions for which normality fails in this way — e.g. the *weak Valiron functions* considered in [BR2] — also belong to the class of Ahlfors islands maps.

The Fatou set $F(f) \subset X$ of an Ahlfors islands map $f : W \to X$ is now defined as usual, except that, in addition to those points which have a neighborhood where all iterates are defined and form a normal family, it will also include all points with $f^j(z) \in X \setminus \overline{W}$ for some j. Thanks to the Ahlfors islands property, Baker's proof of the density of repelling periodic points goes through as usual, and $J(f)$ has all the standard properties.

2.4. Theorem (Baker, Epstein). *Let* $f \in \mathcal{A}(W, X)$. *Then* $J(f)$ *is a nonempty perfect subset of* X, *and repelling periodic points of* f *are dense in* $J(f)$. *Furthermore, if* $J(f) \neq X$, *then* $J(f)$ *has empty interior.*

Note that $J(f) \neq \emptyset$ is trivial by definition as soon as $\partial W \neq \emptyset$, while $\partial W = \emptyset$ means that we are in the well-known case of a rational self-map of the Riemann sphere or of a self-map of the torus. The key fact in the proof of the density of repelling periodic points is the following lemma, which we will use below and whose proof we sketch for completeness. The remaining claims in Theorem 2.4 follow in the usual manner.

2.5. Lemma (Islands near the Julia set). *Let* $f \in \mathcal{A}(W, X)$. *Then there is a number* k *with the following property: if* $V_1, \ldots, V_k \subset X$ *are Jordan domains with disjoint closures, then there is some* V_j *such that every open set* U *with* $U \cap J(f) \neq \emptyset$ *contains an island of* f^n *over* V_j, *for some* n.

Proof. Let $k \geq 5$ be as in the islands property. First suppose that $\partial W \neq \emptyset$; then by the islands property, f has infinitely many islands over some V_j; in particular, we can pick k such islands $G_1, \ldots, G_5$.

If $\partial W = \emptyset$, then f must be either a rational function or a self-map of the torus; in either case, it is easy to see that there is some V_j and k domains $G_1, \ldots, G_k$ which are islands of some *iterates* of f. (For example, this follows by using the fact that $J(f) \neq \emptyset$ and the normal families version of the Ahlfors five-islands theorem.)

Now let U be as in the statement of the lemma. If all iterates of f are defined on U, then the normal families version of the Ahlfors five-islands theorem implies the existence of an island of some iterate of f over one of the domains $G_1, \ldots, G_k$, which is hence also an iterated island over the original domain V_j.

Otherwise, there is some $z_0 \in U$ with $f^n(z_0) \in \partial W$. By shrinking U, if necessary, we may assume that $f^n|_U$ is a proper map with no critical points except possibly z_0. By the islands property, $f^n(U)$ contains an island of f over one of the $G_1, \ldots, G_k$, which then pulls back to an island of f^{n+2} over V_j, as desired. ∎

Remark. Epstein's original definition of finite-type and Ahlfors islands maps allows X to be the disjoint union of several compact Riemann surfaces, in which case the concept of "elementary" cases becomes slightly more involved, but everything else goes through as in the connected case.

For finite-type maps, Epstein proves the absence of Baker and wandering domains as well as the finiteness of nonrepelling cycles. We refer the reader to the thesis [E1] and the manuscript [E2] for further information.

We remark that Oudkerk [O] suggested a slightly modified definition of Ahlfors islands maps which is equivalent to the above, but ensures that, for each k, the class of "Ahlfors k-islands maps" is closed under composition. (With our definition, this is true only for $k \geq 5$.)

Radial Julia sets.

2.6. Definition (Radial Julia sets).
Let $f \in \mathcal{A}(W, X)$. The *radial Julia set* $J_r(f)$ is the set of all points $z \in J(f)$ with the following property: all forward images of z are defined, and there is some $\delta > 0$ such that, for infinitely many $n \in \mathbb{N}$, the disk $\mathbb{D}_\delta(f^n(z))$ can be pulled back univalently along the orbit of z. (I.e.,
$$f : \mathrm{Comp}_z\big(\mathbb{D}_\delta(f^n(z))\big) \to \mathbb{D}_\delta(f^n(z))$$
is a conformal isomorphism.)

Recall that all disks and distances are understood with respect to the metric on the compact surface X (i.e., the spherical metric in the case of rational, entire or meromorphic functions, where $X = \hat{\mathbb{C}}$). We remark that, if $z \in J(f)$ and $\limsup \mathrm{dist}(f^n(z), \mathcal{P}(f)) > 0$, where $\mathcal{P}(f)$ is the postsingular set of f, then $z \in J_r(f)$.

2.7. Lemma and Definition (Disks of Univalence). *Let $z_0 \in J_r(f)$. Then there exists a disk $D = \mathbb{D}_\delta(w_0) \subset X$ with the following properties.*

(a) *Both D and $\mathbb{D}_{2\delta}(w_0)$ are simply connected.*
(b) *for infinitely many n, $f^n(z_0) \in D$ and $\mathbb{D}_{2\delta}(w_0)$ pulls back univalently to z_0 under f^n.*

In particular, if n is as above and U is the component of $f^{-n}(D)$ containing z_0, then

$$\vartheta := \sup_{z\in U} \|Df^n(z)\| < \infty, \quad and \quad U \subset \mathbb{D}_{C/\vartheta}(z_0).$$

(Here C depends only on X.) Furthermore, $\vartheta \to \infty$ as $n \to \infty$.
   We will call such a disk D a disk of univalence *for z_0.*

Proof. Let δ' be the constant from the definition of $J_r(f)$. Let w_0 be a limit point of a sequence $z_n = f^n(z_0)$ of iterates for which $\mathbb{D}_{\delta'}(z_n)$ can be pulled back univalently. Then we set $D := \mathbb{D}_\delta(w_0)$, where $\delta < \delta/2$ is sufficiently small so that D and $\mathbb{D}_{2\delta}(w_0)$ are simply connected.

   The final claim follows from the Koebe distortion theorem. ∎

Hyperbolic dimension. Hyperbolic dimension was introduced by Shishikura in his proof [Sh] that the boundary of the Mandelbrot set has Hausdorff dimension two.

2.8. Definition (Hyperbolic dimension).
Let $f \in \mathcal{A}(W,X)$. A *hyperbolic set* is a compact, forward-invariant set $K \subset W$ such that $\|Df^n|_K\| > \lambda > 1$ for some n and λ.
   The *hyperbolic dimension* of f is the supremum of $\dim_H(K)$ over all hyperbolic sets.

   The following fact is well-known (compare [RvS, Lemma 3.4]).

2.9. Lemma (Hyperbolic sets and the radial Julia set). *Any hyperbolic set is contained in $J_r(f)$. Conversely, any compact and forward invariant subset of $J_r(f)$ is hyperbolic.*
   In particular, the hyperbolic dimension $\dim_{\mathrm{hyp}}(f)$ is no larger than the Hausdorff dimension of $J_r(f)$. □

Conformal iterated function systems. Hyperbolic sets are usually obtained by constructing a conformal iterated function system consisting of iterates of the function f. We review the basic case of the theory for the reader's convenience.

2.10. Definition.
Let $V \subset X$ be simply connected. Also let $U_1, \ldots, U_m \subsetneq V$ be pairwise disjoint simply connected domains with $\overline{U_j} \subset V$, and let $g_j : V \to U_j$ be conformal isomorphisms. Then we say that (g_j) is an *elementary conformal iterated function system*.
   The *limit set* of the system (g_j) is the set

$$L := \bigcap_{n\geq 0} \bigcup_{(j_1,\ldots,j_n)} g_{j_n}(\ldots(g_{j_1}(V))).$$

   (By the Schwarz lemma, the maps g_j are contractions, and hence L is a compact, totally disconnected set.)

In the well-known case where all g_j are affine similarities, the Hausdorff dimension of the limit set is the unique value t with

$$(1) \qquad \sum (\lambda_j)^t = 1,$$

where $|g_j'| \equiv \lambda_j < 1$ [Fa]. We will use the following lemma in the general case.

2.11. Lemma (Hausdorff dimension of limit sets). *Suppose that L is the limit set of a conformal iterated function system (g_j) as above. Set $\lambda_j := \inf_{z \in V} \|Dg_j(z)\|$, and suppose that*

$$\sum \lambda_j{}^t \geq 1.$$

Then $\dim_H(L) \geq t$. $\qquad\qquad \square$

In fact, the Hausdorff dimension of the limit set can be computed exactly by generalizing the formula (1) to the more general setting as follows. We set

$$(2) \qquad P(t) := \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{(j_1, \ldots, j_n)} \|D(g_{j_1} \circ \cdots \circ g_{j_n})(z_0)\|^t \right),$$

where $z_0 \in V$ is some base point. (The existence of the limit and its independence from z_0 follow from the Koebe distortion theorem.)

Then the Hausdorff dimension t is the unique zero of the strictly decreasing function $P(t)$ (usually called the "pressure function"). (For a proof, in the more general setting of *infinite* conformal iterated function systems in any dimension, see [MU].)

We obtain the above lemma as an obvious corollary, since the terms in the limit in (2) are bounded from below by

$$\frac{1}{n} \log \left(\left(\sum \lambda_j^t \right)^n \right) = \log \left(\sum \lambda_j^t \right).$$

2.12. Corollary (Hyperbolic dimension of Ahlfors islands maps). *Let $f \in \mathcal{A}(W, X)$ be a non-elementary Ahlfors islands map. Then $\dim_{\mathrm{hyp}}(f) > 0$.*

Proof. Let k be as in Lemma 2.5, and pick k Jordan domains $U_1, \ldots, U_k$, each of which intersects the Julia set. Then for at least one of these domains, say $U = U_j$, there are islands of some iterate of f over U near every point of the Julia set. Using the fact that $J(f)$ is perfect, we can thus find two (in fact infinitely many) domains $V_1, V_2 \Subset U$, with $\overline{V_1} \cap \overline{V_2} = \emptyset$, such that some iterate of f maps V_j conformally to U.

The corresponding inverse branches $g_j : U \to V_j$ then form an elementary conformal iterated function system, and the limit set L of this system has Hausdorff dimension strictly greater than 0. Clearly L, together with its finitely many forward images is a hyperbolic set for f, and the claim follows. $\qquad\blacksquare$

Hausdorff dimension and backward invariance. We include a proof of the following simple lemma.

2.13. Observation (Hausdorff dimension of backward invariant sets)**.** *Let* $f \in \mathcal{A}(W, X)$ *be a non-elementary Ahlfors islands map, and let* $B \subset X$ *be backward invariant under* f*; i.e.* $f^{-1}(B) \subset B$*.*

If $U \subset X$ *is an open set intersecting* $J(f)$*, then* $\dim_H(B \cap U) = \dim_H(B)$*.*

Proof. It follows easily from the definition of Hausdorff dimension that there is a sequence $D_1, D_2, \ldots$ of Jordan disks with pairwise disjoint closures such that $\dim_H(B \cap D_j) \to \dim_H(B)$.

If U contains some point where f^j is undefined for some $j \geq 0$, then it follows from the Ahlfors islands property that U contains an island over D_j for infinitely many j. Otherwise, this follows from the usual five-islands theorem for (non-)normal families.

So in either case (since conformal maps preserve Hausdorff dimension), we have

$$\dim_H(B \cap U) \geq \dim_H(B \cap D_j)$$

for infinitely many j, which proves the claim. ∎

3. Hyperbolic dimension and radial Julia sets

3.1. Theorem. *Let* $f \in \mathcal{A}(W, X)$ *be any non-elementary Ahlfors islands map. Then*

$$\dim_{hyp}(f) = \dim_H(J_r(f)).$$

Proof. The inequality "$\leq$" follows from Lemma 2.9. So set $d := \dim_H(J_r(f))$ and let $\varepsilon > 0$. We shall show that $\dim_{\mathrm{hyp}}(f) \geq d' := d - \varepsilon$ by constructing a hyperbolic set of dimension at least d'.

Pick a countable basis of the topology of X consisting of simply connected disks D_i. Let $J_r(D_i)$ be the set of points in $J_r(f)$ for which D_i is a disk of univalence as defined in Lemma 2.7. Note that, if $z \in J_r(D_i)$ and $w \in W$ with $f(w) = z$, then either $w \in J_r(D_i)$ or w is a critical point. So $J_r(D_i)$ is the union of a backward invariant set and a countable set, and hence all nonempty open subsets of $J_r(D_i)$ have the same Hausdorff dimension by Observation 2.13.

Since $J_r(f)$ is the countable union of the $J_r(D_i)$, we can find an index i such that $J_r(D_i)$ has Hausdorff dimension greater than d'. Set $D := D_i$ and let $A := J_r(D) \cap D$; then $\dim_H(A) > d'$ by the above. Hence, we can pick $\delta > 0$ such that any covering of A by sets U_j of diameter at most δ has

$$\sum_j \mathrm{diam}(U_j)^{d'} > (10 \cdot C)^{d'},$$

where C is the constant from Lemma 2.7.

Now, for every point $a \in A$, pick a univalent pullback U_a of V around a as in Lemma 2.7 such that

$$\vartheta_a := \|Df^n|_{U_a}\| > \frac{10C}{\delta}$$

and $\overline{U_a} \subset V$. Then A is covered by the open balls

$$\mathbb{D}_{C/\vartheta_a}(a),$$

and by a standard covering theorem (see [M, Theorem 2.1] or [Fe, Theorem 2.84 and corollaries]), we can pick a subsequence a_j such that the balls $\mathbb{D}_{C/\vartheta_j}(a_j)$ are disjoint, (where $\vartheta_j = \vartheta_{a_j}$) while

$$A \subset \bigcup \mathbb{D}_{5C/\vartheta_j}(a_j).$$

By the condition on the size of ϑ_a, these balls have diameter $10C/\vartheta_j < \delta$. So by choice of δ, we have

$$\sum_{j=1}^{\infty} \left(\frac{10C}{\vartheta_j}\right)^{d'} > (10C)^{d'}.$$

If we pick N sufficiently large, set $\lambda_j := 1/\vartheta_j$ and consider the elementary conformal iterated function system given by the branches g_j of f^{-n_j} which take D to U_j, we have

$$\sum_{j=1}^{N} \lambda_j^{d'} > 1,$$

and hence the limit set L of this system satisfies

$$\dim_H(L) \geq d'$$

by Lemma 2.11. We have $L \subset A \subset J_r(f)$ by construction, and L together with finitely many forward images forms a compact, forward invariant set, which is hence a hyperbolic set for f. ∎

APPENDIX: HYPERBOLIC DIMENSION IN LOGARITHMIC TRACTS

If $U \subset \mathbb{C}$ is an unbounded Jordan domain, and $f : U \to \{|z| > R\}$ (where $R > 0$) is a universal covering, then Barański, Karpińska and Zdunik [BKZ] show that there is a hyperbolic set $K \subset U$ for f with $\dim_H(K) > 1$. We remark that this carries over to a slightly more general setting where the asymptotic curves only *accumulate* on the asymptotic value:

A.1. Theorem (Hyperbolic dimension and logarithmic tracts). *Let X be a compact Riemann surface, let $D \subset X$ be a Jordan domain, and let $a \in D$.*
Suppose that $G \subset X \setminus \{a\}$ is a simply connected domain, and that

$$f : G \to D \setminus \{a\}$$

is a universal covering map. Suppose furthermore that there is a sequence (z_n) in G with $z_n \to a$ and $f(z_n) \to a$.
Then G contains a hyperbolic set for f of dimension greater than one. □

In the setting of [BKZ], one has the stronger property that $z_n \to a$ whenever $f(z_n) \to a$ in G. We note that it is easy to construct an Ahlfors function with a logarithmic tract which does not satisfy this stronger condition, but to which the theorem applies. For example, we can consider a function of the form $j \circ \varphi$, where $j : \mathbb{D} \to \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is the elliptic modular function, while $\varphi : \mathbb{D} \to \mathbb{C}$ is a conformal map which takes a suitable boundary point to a nontrivial prime end whose impression contains the asymptotic value ∞.

For a more natural example, suppose that $f : \mathbb{D} \to \mathbb{C}$ is a Valiron function (as discussed in Section 2) which has an asymptotic value over ∞. (Such a function can easily be constructed by the same technique as in [BR1, Example 1].) Then the logarithmic curve itself spirals in to the boundary. Postcomposing f with a suitable Möbius transformation which takes ∞ to the unit circle gives rise to an Ahlfors islands map with the aforementioned property.

The proof of our more general version is analogous to that in [BKZ]; the underlying ideas can be traced to Stallard's original argument from [S2]. We will briefly explain one step in the proof which differs slightly in our setting.

Consider an annulus A centered at a (in some fixed local coordinates), of sufficiently large modulus and sufficiently close to a. Then, in the setting of [BKZ], the preimage of this annulus under f is an infinite strip S tending to a in both directions. By basic estimates on the hyperbolic metric, S must contain points on the outside of the annulus, and hence crosses A. This fact allows Barański, Karpińska and Zdunik to construct an iterated function system (in logarithmic coordinates) whose limit set has Haudorff dimension greater than one.

With the more general assumptions in Theorem A.1, it is no longer true that the strip S must tend to a. (Recall the examples discussed above.) However, S still crosses the annulus A at least once, and possibly several times. (This follows from the assumptions on f and well-known results of geometric function theory, e.g. the fact that radial limit points of a conformal map are dense in the boundary of its image.) The remainder of the proof goes through exactly as in [BKZ]; we refer the reader to this article for further details.

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